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Traces in braided categories

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Abstract

With any even Hecke symmetry R (that is a Hecke type solution of the Yang–Baxter equation) we associate a quasitensor category. We formulate a condition on R implying that the constructed category is rigid and its commutativity isomorphisms $R_{U,V}$ are natural in the sense of [20]. We show that this condition leads to rescaling the initial Hecke symmetry. We suggest a new way of introducing traces as properly normalized categorical morphisms $\mathrm{End}(V) \to \mathbb{K}$ and deduce the corresponding normalization from categorical dimensions.
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1. Introduction

The main purpose of this paper is to introduce some braided categories arising from "nonquasiclassical" Hecke symmetries constructed in [10] and to suggest a way of defining categorical traces

$$\mathrm{tr} = \mathrm{tr}_V : \mathrm{End}(V) \to \mathbb{K}, \quad V \in \mathrm{Ob}(\mathcal{C}) \tag{1.1}$$

in a somewhat elementary way without using any ribbon element (see below). Hereafter $\mathrm{Ob}(\mathcal{C})$ stands for the set of objects of the category $\mathcal{C}$ and $\mathbb{K}$ stands for the basic field, always $\mathbb{C}$ or $\mathbb{R}$.

Let us precise from the very beginning that by braided categories we mean monoidal tensor or quasitensor ones whose objects are vector spaces with the usual tensor product

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and whose natural associativity isomorphisms are identical (for the terminology, cf., e.g., [3,20]). Thus, the structure of such a category is principally determined by commutativity isomorphisms

$$R_{U,V} : U \otimes V \to V \otimes U, \quad U, V \in \mathrm{Ob}(\mathcal{C}). \tag{1.2}$$

These isomorphisms are assumed to satisfy the so-called Yang–Baxter (YB) equation. They play the role of the usual flip which transposes the factors. We call the isomorphisms (1.2) YB operator, braiding, or quantum R-matrix.

Moreover, we assume them to be natural in the sense of [20]. This means that for any two categorical morphisms $f : U \to U'$ and $g : V \to V'$ one has:

$$(g \otimes f) \circ R_{U,V} = R_{U',V'} \circ (f \otimes g). \tag{1.3}$$

We show that this condition leads to a normalization of the commutativity isomorphisms different from that (1.8) usually employed for Hecke type braidings. (Let us note that even in the case of the quantum group $U_q(sl(n))$ the YB operators coming from the universal R-matrix must be rescaled.)

In what follows we identify $\mathbb{K}$ and the $\mathrm{End}(I)$, where I is the identity object in the sense of [3]. Moreover, we use the notation $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ for the objects of the category which sometimes are called internal (endo)morphisms (cf. [4]). By contrast, the morphisms of the category in question will be called "categorical morphisms". For example, categorical morphisms of the category $\mathbf{g}-\mathrm{Mod}$ of modules over a Lie algebra $\mathbf{g}$ are elements of $\mathrm{Hom}(U, V)$, $U, V \in \mathrm{Ob}(\mathbf{g} - \mathrm{Mod})$, commuting with the action of $\mathbf{g}$.

Initially, categories equipped with commutativity morphisms were introduced by MacLane [17]. However, he only considered involutary YB operators ($R^2 = \mathrm{id}$) called in what follows "symmetries". A new wave of interest in braided categories (but with noninvolutary braidings) arose in connection with integrable system theory. More precisely, such categories appeared as those of modules of Drinfeld–Jimbo quantum groups (QG) $U_q(\mathbf{g})$ playing an important role in this theory (we will denote this category $U_q(g) - \mathrm{Mod}$). Besides, the QG have found many other interesting applications, in particular, in noncommutative geometry.

The problem we consider in the paper is a categorical definition of the trace (1.1). What is a reasonable generalization of the basic property of the classical trace

$$\mathrm{tr}[X, Y] = 0 \quad \forall X, Y \in \mathrm{End}(V), \tag{1.4}$$

where $[X, Y]$ is the commutator of two endomorphisms?[1]

If the braiding in a given braided category is involutary and invertible by column in the sense of the formula (2.48) then there exists a natural generalization of the above commutator such that relation (1.4) is still valid for this commutator and an appropriate trace. For example

[1] Sometimes this property is attributed to the quantum trace (cf. [14]). Equivalent form of this property $\mathrm{tr}(gXg^{-1}) = \mathrm{tr}\, X$ ($g \in \mathrm{End}(V)$ is assumed to be invertible) appeared in [20]. However, all this is true if we assume X, Y and g to be categorical morphisms. But in this case the conditions above become meaningless. It is easy to see by taking a simple object V, since it only admits scalar categorical morphisms and these conditions degenerate.

it is so for a super-commutator and super-trace defined in a super-category. Other examples can be found in [10].[2]

Let us note that the definition of the trace (1.1) in a category equipped with a symmetry makes use of this symmetry and of the identification $\mathrm{End}(V) = V \otimes V^*$, where V^* is an object dual to V (all categories in question are assumed to be rigid, i.e., for any object of the category its dual is also an object). Up to our knowledge for the first time such a trace has been introduced in [6].

Once such a trace is defined one can introduce *a categorical dimension* $\dim(V)$ of an object V by setting $\dim(V) = \mathrm{tr}(\mathrm{id})$. It is easy to see that

$$\dim(U \oplus V) = \dim(U) + \dim(V), \qquad \dim(U \otimes V) = \dim(U)\dim(V), \tag{1.5}$$

i.e., the dimension can be considered as an additive and multiplicative (a–m) functional on the objects of the category.

However, a direct application of the approach of [6] to the category $U_q(\mathbf{g}) - \mathrm{Mod}$ gives "dimension" which is not an a–m functional. This is the reason why one has to introduce a correction in the definition of "dimensions" turning them into an a–m functional. The correction is connected to the so-called ribbon element in the corresponding QG. The image of this element is sometimes called "twist" (cf. [20]). The categories possessing a braiding and a twist are called ribbon. The trace in such categories is defined via some combination of the ribbon element and another element u due to Drinfeld (cf. [3]) and this leads to the dimension which is an a–m functional.

Some generalization of this construction of the trace has been suggested in [2]. Suggested in that paper is a way to define the notion of dimension in some categories without any braiding. Such a category is introduced as that of H-modules, where H is a Hopf algebra. Instead of a universal quantum R-matrix (i.e., the element giving rise to a braiding) the authors of [2] use another element allowing to identify any object V with its second dual V^{**}.

In this, paper we restrict ourselves to categories equipped with braidings and suggest another somewhat elementary way of introducing traces.

Before discussing this way let us introduce some notations and definitions.

A Yang–Baxter operator

$$R : V^{\otimes 2} \to V^{\otimes 2} \tag{1.6}$$

will be called *a Hecke symmetry* if in addition to the Yang–Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \tag{1.7}$$

it also satisfies the relation

$$(q\,\mathrm{id} - R)(q^{-1}\,\mathrm{id} + R) = 0, \quad q \in \mathbb{K}. \tag{1.8}$$

Here the standard tensor notations have been used for the equations and operators in tensor product of spaces. Eq. (1.7) is written in $V^{\otimes 3}$ and one assumes

$$R_{12} = R \otimes \mathrm{id}, \qquad R_{23} = \mathrm{id} \otimes R,$$

where id is the identity operator on V.

[2] In this case the trace can be treated in terms of "R-cyclic cohomology" which can be naturally defined via the operator R (in the spirit of super-cyclic cohomology). However, apparently there does not exist any reasonable similar treatment of the trace in the categories under consideration.

The nonzero parameter q is taken to be generic, which means it is not a root of unity: $\forall k \in \mathbb{N}, k \neq 1, q^k \neq 1$. As a consequence none of the so-called *q-numbers* k_q is equal to zero

$$k_q \equiv \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \quad \forall k \in \mathbb{N}. \tag{1.9}$$

Let us assign to such a Hecke symmetry "symmetric" $\Lambda_+(V)$ and "skew-symmetric" $\Lambda_-(V)$ algebras by

$$\Lambda_+(V) = \frac{T(V)}{\{\mathrm{Im}(q\,\mathrm{id} - R)\}}, \qquad \Lambda_-(V) = \frac{T(V)}{\{\mathrm{Im}(q^{-1}\,\mathrm{id} + R)\}}. \tag{1.10}$$

Here $T(V)$ stands for the free tensor algebra of the space V and $\{I\}$ denotes the ideal generated by a subset I in a given algebra. Let $\Lambda^l_\pm(V)$ be a homogeneous component of the algebra $\Lambda_\pm(V)$ of the degree l. We call a Hecke symmetry *even* if there exists an integer p such that the component $\Lambda^l_-(V)$ is trivial at $l > p$ and it is one-dimensional at $l = p$. The integer p will be called a rank of V and be denoted $\mathrm{rk}(V)$.

In the latter 1980s one of the authors (DG) constructed examples of Hecke type braidings which differ drastically from those related to the QG (see [10] and references therein). Namely, it was shown that there exist a lot of Hecke symmetries R such that $n = \dim(V) > p = \mathrm{rk}(V)$. Here by $\dim(V)$ we mean the usual dimension of the space V. Note that these Hecke symmetries are not deformations of the usual flip and we call them nonquasiclassical.[3] To the contrary, the Hecke symmetry coming from the QG $U_q(sl(n))$ is a deformation of the flip and we call it quasiclassical.

In this paper, we introduce a braided category generated by a vector space V equipped with a Hecke symmetry such that its braidings are natural. We call it the Schur–Weyl (SW) category and denote as SW(V). Let us remark that similar categories were considered in some papers (cf. [1,14,15]). However, if the authors of [15] "reconstruct" an existing category we have no category at the very beginning and should first construct it.

We construct the category directly by giving the list of objects and categorical morphisms without using any RTT algebra habitually employed for that ([1,14]). This leads to different defining morphisms of the category and finally, to the condition (2.33) ensuring naturality of the braidings $R_{U,V}$. (In a separate publication one of us (RL) shows that in the class of nonquasiclassical Hecke symmetries from [10] there exists a big subclass of those satisfying this condition.)

Now let us describe the category under consideration. Any object of such a category is a direct sum of simple ones and in this sense the category SW(V) is spanned by simple (basic) objects. These basic objects are labeled by partitions (or what is the same by Young diagrams)

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}), \quad \lambda_{i+1} \leq \lambda_i, \tag{1.11}$$

λ_i being nonnegative integers. In the sequel we will use the notation $\lambda \vdash k$ for each λ being a partition of the integer k that is $\sum_i \lambda_i = k$. The number of nonzero components of a

[3] Some of them have been independently introduced in [7].

partition λ is called its *height* and will be denoted $\ell(\lambda)$. For an object corresponding to a diagram λ we introduce the notation V_λ.

Let us stress that the tensor product of two basic objects is a direct sum of basic ones. This naturally leads us to the notion of Grothendieck semiring (and hence ring) of the category which turns out to be the same as for the categories of $sl(n)$- (or $U_q(sl(n))$)-modules. The only difference is that the role of the classical dimension $n = \dim(V)$ is played by the rank $p = \mathrm{rk}(V)$. Thus, we can consider the dimensions of objects of the category as a functional on the Grothendieck semiring. Since it is an a–m functional we come to the problem of describing the whole family of a–m functionals and select those related to traces.

By using the standard technique of symmetric functions (cf. [16]) we show in Proposition 15 that such a functional f is determined by its values on objects $V_{(1^k)}$, $k = 1, 2, \ldots, p-1$, and its value on a basic object V_λ is equal to

$$f(V_\lambda) = s_\lambda(\alpha_1, \alpha_2, \ldots, \alpha_p),$$

where s_λ is the Schur function (polynomial) in p variables corresponding to the partition λ. The numbers $(-\alpha_i)$ are the roots of the polynomial

$$\phi(t) = t^p + f_1 t^{p-1} + \cdots + f_{p-1} t + 1, \qquad f_k = f(V_{(1^k)}).$$

(Note, that the classical (usual) dimension is also an a–m functional. Therefore, the above result allows one to calculate the classical dimension of spaces V_λ provided that the Poincaré series $P_-(t)$ (see Remark 19) is known (for involutary symmetries such a calculation has been done in [11]). Remark, that if a Hecke symmetry $R(q)$ (1.6) is a deformation of an involutary one $R(1)$ then the usual dimension of corresponding spaces V_λ will be the same in both cases. Indeed, being an integer number such a dimension is stable under deformation, while the categorical dimension is a function in q which becomes integer at $q = 1$.)

Having classified all a–m functionals one can put the question: which a–m functionals come from SW categories? More precisely, for which a–m functional on a given Grothendieck semiring (with a fixed p) there exists an SW category such that the categorical dimension of its objects coincides with this functional?

Now we are going back to the problem of defining traces on objects $\mathrm{End}(U) = U \otimes U^*$ (for the sake of concreteness we consider algebras of left endomorphisms). Let us assume that trace is defined on $\mathrm{End}(U \oplus V) = (U \oplus V) \otimes (U^* \oplus V^*)$ via

$$\mathrm{tr}|_{\mathrm{End}(U)\oplus\mathrm{End}(V)} = \mathrm{tr}|_{\mathrm{End}(U)} + \mathrm{tr}|_{\mathrm{End}(V)}$$

being extended to $U \otimes V^* \oplus V \otimes U^*$ by 0. Then traces are completely defined by their values on simple objects. If V is such an object then the trivial component in the product $V \otimes V^*$ is unique. So, the morphism (1.1) for such V (being nothing but a multiple of the projection on the trivial component) is uniquely defined up to a nontrivial factor.

As a trace we take this properly normalized morphism. Our "proper" normalization is fixed by the requirement that the corresponding dimension should be an a–m functional on the corresponding Grothendieck semiring. Thus, this normalization is a collective phenomenon.

Remark that our method to define traces as properly normalized categorical morphisms is valid, in principle, for any category for which the Grothendieck semiring is well defined.

This method does not make use of either ribbon element or twist. It is sometimes useful to employ twists in order to calculate categorical dimensions (as it is done in [14]) but it is not reasonable to introduce it in the definition of traces. Also, it turns out that this method is useful for studying K-theory of some algebras related to SW categories and computing noncommutative index for them. This application will be presented elsewhere.

As for the category under consideration a computation shows that the corresponding dimensions of basic objects are

$$\dim_q(V_\lambda) = s_\lambda(q^{p-1}, q^{p-3}, \ldots, q^{-(p-1)}). \tag{1.12}$$

In a little bit different form the formula (1.12) was also given in [14,15]. (Also remark that in the case of the category $U_q(sl(n)) - \mathrm{Mod}$ the above formula is equivalent to that from [9].)

However, behind this nondetailed information about the objects encoded in their dimensions different and rich structures are hidden. For example, traces (1.1) in the categories in question are defined via some matrix C entering the formula for the trace on the basic space V and extended in a proper way onto the whole category (the extension of C matrix to the space V_λ will be denoted C_λ). It is worth mentioning that the matrices C_λ are essentially different in quasiclassical and nonquasiclassical cases while the categorical dimensions of objects which are nothing but the properly normalized usual traces of C_λ depend only on λ and $p = \mathrm{rk}(V)$ via the formula (1.12).

The paper is organized as follows. In Section 2 we construct all the necessary elements of an SW category: the class of objects, categorical morphisms, the quasitensor and rigid structure. At the end of Section 2 we discuss the role of the condition (2.33) for applications. In Section 3 we describe the full set of a–m functionals on the objects of SW(V), define the trace in End(U) $\forall\, U \in \mathrm{Ob}(\mathrm{SW(V)})$, and calculate $\dim_q U$ as the value of trace on identity morphism in End(U).

2. SW category generated by a Hecke symmetry

In this section, we describe the construction of SW category SW(V) (see Section 1) generated by a finite-dimensional vector space V equipped with a Hecke symmetry. We supply the category with attributes of that $U_q(sl(n)) - \mathrm{Mod}$. In particular, all categories SW(V) are $\mathbb{K}$-linear rigid quasitensor ones (for precise definitions see [3,18]). As was shortly outlined in Section 1, the basic objects of SW(V) are some linear subspaces V_λ of the tensor algebra $T(V)$, while the categorical morphisms will be specific linear mappings of the objects (see below). Consequently, a $\mathbb{K}$-linear structure of such a category is obvious: as the direct sum of objects we will take the usual direct sum of linear spaces and the trivial (zero) vector space will be the null object of the category. So, we should only define the structure of monoidal category and fix braiding isomorphisms converting the monoidal structure into the quasitensor (or braided) one. Besides, to have a rigid category, we must ensure that the dual space V^* of an arbitrary $V \in \mathrm{Ob}(\mathrm{SW(V)})$ is also an object of the category. Now we pass to the explicit construction of the mentioned components of the category SW(V) taking into account condition (1.3).

2.1. Class of objects

Let us fix some finite-dimensional vector space V, $\dim V = n$, and choose a basis $\{e_i\}$ in this space. Consider a linear operator R (1.6) whose action on the basis elements reads:

$$R(e_i \otimes e_j) = e_r \otimes e_s \, R_{ij}^{rs},$$

where the summation over the repeated indices is understood. In what follows we assume R to be a Hecke symmetry with a generic $q \in \mathbb{K}$.

Besides, we will suppose R to be an even symmetry of a rank $p \leq n = \dim V$. This means that the Poincaré series $P_-(t)$ related to the "skew-symmetric" algebra $\wedge_-(V)$ (1.10) is a monic polynomial of pth degree, that is a polynomial in t whose leading coefficient is equal to 1.

Throughout the paper we will use the compact notation for the relations including matrices and vector spaces. In such notations the indices will stand for a number of space rather than for a particular matrix or vector component. A basis vector e_i in the kth matrix space is denoted as $e_{\langle k|}$, while for the basis vector of dual space we will write $e^{|k\rangle}$. The same principle is applied to components of arbitrary tensors.

For example, the above relation will look like[4]

$$R(e_{\langle 1|} \otimes e_{\langle 2|}) := e_{\langle 1|} \otimes e_{\langle 2|} \, R_{\langle 12|}^{|12\rangle} := e_{\langle 1|} \otimes e_{\langle 2|} \, R_{12}.$$

In this notation the summation over the repeated indices is represented by an expression which contains the same upper and lower case indices *with properly oriented brackets* as shown in the examples below:

$$e_{\langle 1|} \otimes e^{|1\rangle} := \sum_i e_i \otimes e^i, \qquad e_{\langle 1|} \cdot T_1 := e_{\langle 1|} \, T_{\langle 1|}^{|1\rangle} := \sum_j e_j T_i^j,$$

$$M_1 R_{12} N_2 := \sum_{a,b} M_a^{i_1} R_{j_1 b}^{a i_2} N_{j_2}^b$$

but

$$e^{|1\rangle} \otimes e_{\langle 1|} := e^i \otimes e_j, \qquad e_{\langle 1|} \otimes e^{|2\rangle} := e_i \otimes e^j, \qquad T_1 \cdot e_{\langle 1|} := T_j^i \, e_k,$$

and so on.

Now we use the fact, that in each homogeneous component $V^{\otimes m}$ of the tensor algebra $T(V)$ one can realize the so-called *local representation* of the Hecke algebra H_m [5] via a given Hecke symmetry.[5]

The Hecke algebra H_m is generated by the unit element id_H and $m-1$ generators σ_k subject to the following relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \ \ |i-j| \geq 2,$$
$$(\sigma_i - q \, \mathrm{id}_H)(\sigma_i + q^{-1} \, \mathrm{id}_H) = 0, \quad i = 1, 2, \ldots, m-1.$$

[4] The symbol := means "by definition".

[5] For the review on Hecke algebra the reader is referred to the recent work in [19].

The local representation of H_m in $V^{\otimes m}$ is of the form:

$$\sigma_i \to \rho_R(\sigma_i) \equiv R_{ii+1} = \mathrm{id}_{i-1} \otimes R \otimes \mathrm{id}_{m-i-1} \in \mathrm{End}(V^{\otimes m}). \tag{2.1}$$

In the further construction of SW(V) category the central role belongs to the fact that the Hecke algebra H_m for a generic q is semisimple and can be decomposed into a direct sum of ideals. Moreover, since the Hecke algebra H_m at a generic q is isomorphic to the group algebra $\mathbb{K}[S_m]$ of the mth-order permutation group S_m, its primitive idempotents generating the ideals in mentioned decomposition can be put into one-to-one correspondence with the set of all standard Young tableaux connected with each possible partition λ of the integer m. Speaking more explicitly, given the algebra H_m and a partition $\lambda \vdash m$, one can construct some polynomials Y^λ_{ii} in generators $\{\sigma_i\}$ which turn out to be the primitive idempotents of Hecke algebra H_m (for detailed description of such a construction see review [19]):

$$\mathrm{id}_H = \sum_{\lambda \vdash m} \sum_{i=1}^{d_\lambda} Y^\lambda_{ii}, \tag{2.2}$$

$$Y^\lambda_{ii}\, Y^\mu_{jj} = \delta_{ij}\delta^{\lambda\mu}\, Y^\lambda_{ii}. \tag{2.3}$$

In the above formulas the index i runs from 1 to $d_\lambda \equiv \dim \lambda$, that is to the number of all standard Young tableaux, corresponding to the Young diagram of the partition $\lambda \vdash m$. Recall, that a Young tableau is called *standard* if it is filled with successive integers from 1 to m in such a way that the integers increase from left to right in each row and from top to bottom in each column. The standard Young tableaux (and hence the primitive idempotents Y^λ_{ii}) may be lexicographically ordered in many ways and for definiteness we fix an order in which the first idempotent Y^λ_{11} corresponds to the Young tableau filled by integers consequently increasing by 1 when going down in each column. Here is an example for the Young tableau corresponding to the partition $\lambda = (3, 2^2, 1)$:

$$Y^\lambda_{11} \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 \\ \cline{1-2} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\ . \tag{2.4}$$

A primitive idempotent corresponding to the partition $\lambda = (1^m)$ will be called the antisymmetrizer and be denoted as $A^{(m)}$. There is one important circumstance about the images of $A^{(m)}$ in $\mathrm{End}(V^{\otimes m})$ with respect to representation (2.1). Since the YB operator R is taken to be an even Hecke symmetry of rank p, the image of the pth-order antisymmetrizer

$$\mathcal{A}^{(p)} := \rho_R(A^{(p)}) : V^{\otimes p} \to V^{\otimes p}$$

is one-dimensional. So, the action of this projector on an arbitrary basis element of $V^{\otimes p}$ can be presented in the form:

$$\mathcal{A}^{(p)} e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} = e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|}\, v^{|1\cdots p\rangle} u_{\langle 1\cdots p|}, \tag{2.5}$$

where components of the tensors u and v belong to the field $\mathbb{K}$. Due to (2.3) the normalization of u and v is fixed to be $u_{\langle 1\cdots p|} v^{|1\cdots p\rangle} = 1$. The images of antisymmetrizers $A^{(m)}$ with $m > p$

are trivial:

$$\operatorname{Im} \rho_R(A^{(m)}) = 0 \quad \forall m \geq p+1. \tag{2.6}$$

In accordance with general theory each primitive idempotent Y_{ii}^{λ} generates a left or right ideal of the Hecke algebra H_m by means of left or right multiplications on all possible elements of H_m. Equivalently, one can consider the left or right *regular module* over H_m which as a vector space is H_m itself and left or right action consists in the left or right multiplication by Hecke algebra elements. In these terms the left or right ideals generated by primitive idempotents will be irreducible left or right H_m submodules. Let us denote left and right submodules generated by Y_{ii}^{λ} as $M_{\lambda}^{l}(i)$ and $M_{\lambda}^{r}(i)$, respectively. As a direct consequence of (2.2) one can write the following decomposition:

$$H_m = \bigoplus_{\lambda \vdash m} \bigoplus_{i=1}^{d} M_{\lambda}^{r}(i), \tag{2.7}$$

and a similar decomposition holds for left submodules $M_{\lambda}^{l}(i)$.

Consider in more detail the structure of the component of (2.7) corresponding to a fixed partition $\lambda \vdash m$. Let us denote

$$\bar{M}_{\lambda} \equiv \bigoplus_{i=1}^{d} M_{\lambda}^{r}(i) \equiv \bigoplus_{i=1}^{d} M_{\lambda}^{l}(i). \tag{2.8}$$

The set $\bar{M}_{\lambda}$ is a two-sided submodule in H_m. Let us dwell upon the question of linear basis in $\bar{M}_{\lambda}$. As is known from the theory of Hecke algebra, $\bar{M}_{\lambda}$ is isomorphic to the algebra $\mathrm{Mat}_{d_{\lambda}}(\mathbb{K})$ of $d_{\lambda} \times d_{\lambda}$ matrices. Therefore in $\bar{M}_{\lambda}$ one can find the set of d_{λ}^2 linear independent quantities Y_{ij}^{λ}, which are some polynomials in generators σ_i of H_m obtained from the primitive idempotents Y_{ii}^{λ}. These quantities form a linear basis in $\bar{M}_{\lambda}$ and obey the algebra of matrix units[6] E_{ij} which constitute the linear basis in $\mathrm{Mat}_{d_{\lambda}}(\mathbb{K})$:

$$Y_{ij}^{\lambda} Y_{kl}^{\lambda} = \delta_{jk} Y_{il}^{\lambda}, \quad i, j, k, l = 1, \ldots, d_{\lambda}. \tag{2.9}$$

Note, that for each fixed i the elements Y_{ij}^{λ} $1 \leq j \leq d_{\lambda}$ form a linear basis in the irreducible submodule $M_{\lambda}^{r}(i) \subset \bar{M}_{\lambda}$ generated by the primitive idempotent Y_{ii}^{λ}. And on the other hand, for each fixed j the elements Y_{ij}^{λ} $1 \leq i \leq d_{\lambda}$ form a linear basis in $M_{\lambda}^{l}(j)$ generated by Y_{jj}^{λ}. Therefore we have $\dim M_{\lambda}^{r}(i) = \dim M_{\lambda}^{l}(i) = d_{\lambda}$.

Otherwise stated, if we arrange the elements Y_{ij}^{λ} into a rectangular $d_{\lambda} \times d_{\lambda}$ matrix, then its rows will represent the linear basises of d_{λ} right submodules $M_{\lambda}^{r}(i)$, while the columns of the matrix will represent the linear basises of left submodules $M_{\lambda}^{l}(i)$. The diagonal entries of the matrix are the primitive idempotents Y_{ii}^{λ}.

An important property of Y_{ii}^{λ} consists in the following fact. For any two idempotents Y_{ii}^{λ} and Y_{jj}^{λ} corresponding to the same partition λ there exists an invertible element of H_m which transforms one of these idempotents into the other one. This means that the submodules $M_{\lambda}^{r}(i)$ (or $M_{\lambda}^{l}(i)$) with different i are isomorphic: they can be transformed into each other

[6] Matrix unit E_{ij} $1 \leq i, j \leq m$ of the matrix algebra $\mathrm{Mat}_m(\mathbb{K})$ is the $m \times m$ matrix with the only nonzero entry at the intersection of the ith row and the jth column which is equal to 1.

by *left* (respectively *right*) multiplication on some invertible element of Hecke algebra H_m. Note, that the whole $\bar{M}_\lambda$, being a two-sided submodule in H_m, is invariant with respect to such a multiplication.

In what follows we will not distinguish the isomorphic submodules corresponding to a partition λ and will consider them up to an isomorphism. Then each submodule $M^r_\lambda(i) \subset \bar{M}_\lambda$ generated by Y^λ_{ii} can be treated as the image of an irreducible H_m module M^r_λ (labeled only by the partition $\lambda \vdash m$) w.r.t. the following H_m module monomorphism:

$$M^r_\lambda \to M^r_\lambda(i) \subset \bar{M}_\lambda \subset H_m, \quad \lambda \vdash m. \tag{2.10}$$

In this sense the submodule $\bar{M}_\lambda$ is isomorphic (as the H_m module) to a direct sum of d_λ copies of the module M^r_λ:

$$\bar{M}_\lambda \equiv \bigoplus_{i=1}^{d_\lambda} M^r_\lambda(i) \cong (M^r_\lambda)^{\oplus d_\lambda}.$$

This formula reflects the well-known fact that the regular representation of a finite-dimensional semisimple algebra decomposes into a direct sum of irreducible modules and the multiplicity of each module is equal to its dimension.

Remark 1. It is worth mentioning that the module M^r_λ can be mapped into $\bar{M}_\lambda$ in many different ways and submodules $M^r_\lambda(i)$ (generated by a fixed choice of primitive idempotents $\{Y^\lambda_{ii}\}$) are only particular cases of all possible monomorphisms. Indeed, let us take an arbitrary set of orthonormal $d_\lambda \times d_\lambda$ projectors $P^{(i)}$ of the rank 1:

$$P^{(i)} \cdot P^{(j)} = \delta^{ij} P^{(i)}, \qquad \sum_{i=1}^{d_\lambda} P^{(i)} = \mathrm{id}.$$

Then, as is evident from above relations and (2.9), the quantities $X^\lambda_{ii} = \sum_{r,s} P^{(i)}_{rs} Y^\lambda_{rs}$ are also a set of primitive idempotents in H_m leading to another decomposition of $\bar{M}_\lambda$ into a direct sum of right submodules which will represent other possible monomorphisms of M^r_λ into $\bar{M}_\lambda$.

Now with each right[7] submodule $M^r_\lambda(i) \subset H_m$ we can associate a space $V_\lambda(i) \subset V^{\otimes m}$ in the following way:

$$V_\lambda(i) = \mathrm{Im}\, \rho_R(M^r_\lambda(i)). \tag{2.11}$$

Besides, we will deal with a space $\bar{V}_\lambda$ which is the image of $\bar{M}_\lambda$ (2.8):

$$\bar{V}_\lambda = \mathrm{Im}\, \rho_R(\bar{M}_\lambda) \subset V^{\otimes m}. \tag{2.12}$$

The spaces $V_\lambda(i)$ with different i (and all other $V_\lambda(X) \in \bar{V}_\lambda$ which are ρ_R-images of other possible monomorphisms[8] of M^r_λ into $\bar{M}_\lambda$) are isomorphic as vector spaces and we will not

[7] The choice of *right* submodules is made since we prefer to use the *left* action of $T \in \mathrm{End}(V)$ on V. Since an arbitrary element of $M^r_\lambda(i)$ has the form $Y^\lambda_{ii} f(\sigma_1, \ldots, \sigma_m)$ (f being a polynomial in the generators of H_m), then with such a choice the projector $\rho_R(Y^\lambda_{ii})$ will be the last element acting on $V^{\otimes m}$ in formula (2.11).

[8] See Remark 1.

distinguish them. Instead, we will deal with a space V_λ which (like M^r_λ) gives rise to a class of isomorphic embeddings $V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes m}$ and each $V_\lambda(i)$ (or any other $V_\lambda(X)$ as well) is just a particular representative of this class of isomorphic spaces.

Remark 2. Formula (2.11) has a nontrivial meaning for $m \geq 1$, where at $m = 1$ the only space V_λ is the space V itself. For the future convenience we extend the formula to the case $k = 0$. Namely, we will take by definition $V_{\lambda\vdash 0} = V_0 \equiv \mathbb{K}$.

Now we can define the class of objects of the SW(V) category generated by a finite-dimensional space V over a field $\mathbb{K}$ equipped with a Hecke symmetry R (1.6)–(1.8).

Definition 3. To each fixed nonnegative integer $k \in \mathbb{Z}_+$ (for $k = 0$ see Remark 2) and each possible partition $\lambda \vdash k$ we put into correspondence a space V_λ isomorphic to any $V_\lambda(i)$ in (2.11). The spaces V_λ, $\lambda \vdash k$, $k \in \mathbb{Z}_+$ are the basic objects of the category SW(V). The whole class of objects of the category is formed by direct sums of a finite number of basic objects V_λ. Thus, the spaces $V_\lambda(i)$ can be treated as the space V_λ equipped with an embedding $\{V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes k}\}$.

Remark 4. The objects V_λ should be simple objects of our category. However, some objects V_λ of the category will be identified with each other (in particular, $V_{(1^p)}$ and V_0). So, finally simple objects of our category will be V_λ modulo the mentioned identification. Up to this identification we will sometimes use the notation $[V_\lambda]_k$ for the family of embeddings $\{V_\lambda \hookrightarrow \bar{V}_\lambda \subset V^{\otimes k}\}$.

Let us turn now to the definition of morphisms of our category.

2.2. *Morphisms of the first kind and the structure of quasitensor category*

Let us denote $\mathrm{Mor}(U, V)$ the space of categorical morphisms $U \to V$.

Definition 5. The morphisms of the first kind are defined as follows:

i) The set $\mathrm{Mor}(V_\lambda, V_\lambda)$ for any basic object V_λ contains only multiples of the identical morphism. That is for any morphism $f : V_\lambda \to V_\lambda$ we have by definition $f = a\,\mathrm{id}$ for some $a \in \mathbb{K}$.
ii) For an object embedded in $V^{\otimes k}$ the morphisms of the first kind are represented by a set of linear mappings:

$$\forall k \in \mathbb{N},\ \ \forall \tau \in H_k, \qquad \phi^k(\tau) = \rho_R(\tau) : \quad V^{\otimes k} \to V^{\otimes k}. \tag{2.13}$$

Thus, being restricted to the set of subspaces $[V_\lambda]_k$ such a mapping ϕ^k sends each $V_\lambda(i) \in [V_\lambda]_k$ to an isomorphic space or to zero space.[9] This means that nontrivial mappings ϕ^k at most change the embedding of V_λ into $V^{\otimes k}$ and therefore are multiples of identical morphism for the basic objects. Among the morphisms ϕ^k there is no one which

[9] For example, $\rho_R(\sigma_i + q^{-1}\,\mathrm{id}_H)$ $1 \leq i \leq k-1$ sends to zero the image of the antisymmetrizer $A^{(k)}$.

would send V_λ to V_μ with $\lambda \neq \mu$. A different kind of morphisms is considered in Section 2.3.

Our next step consists in constructing a monoidal structure, which allows us to "multiply" the objects of the category. This means that we want to define a covariant functor

$$\otimes : \mathrm{Ob(SW)} \times \mathrm{Ob(SW)} \to \mathrm{Ob(SW)}$$

with some associativity morphisms (for detail see [3,18]). As such a functor we take the usual tensor product of linear spaces with associativity morphisms to be identical. So, we have to prove that the tensor product of any two objects of our category is also an object, that is it can be decomposed into a direct sum of basic objects. Evidently, one only needs to verify this property for the tensor product of basic objects V_λ.

Proposition 6. *For given $\lambda \vdash n$ and $\mu \vdash m$ the tensor product of two basic objects V_λ and V_μ can be expanded into a direct sum of basic objects V_ν, $\nu \vdash (n+m)$:*

$$V_\lambda \otimes V_\mu = c^\nu_{\lambda\mu} V_\nu$$

the coefficients $c^\nu_{\lambda\mu}$ being the Littlewood–Richardson ones entering the formula for product of Schur symmetric functions s_λ in p variables.[10]

Proof. As is clear from the definition of basic objects V_λ of our category the structure of their tensor product is controlled by that of modules M^r_λ (2.10) and the assertion of the proposition can be reformulated in terms of these modules.

To do so let us consider an embedding of $V_\lambda \otimes V_\mu$ into $V^{\otimes(n+m)}$ of the form

$$V_\lambda \otimes V_\mu \to V_\lambda(i) \otimes V_\mu(j) \subset V^{\otimes(n+m)}$$

for some fixed i and j. As was defined in (2.11), the space $V_\lambda(i)$ is the image of the right H_n submodule $M^r_\lambda(i)$ under representation ρ_R (2.1).

Consider an embedding $\mathcal{J} : H_n \times H_m \hookrightarrow H_{n+m}$ which on generators looks as follows:

$$\begin{aligned} &\forall \sigma'_i \in H_n : \mathcal{J}(\sigma'_i) = \sigma_i \in H_{n+m}, \quad 1 \le i \le n-1, \\ &\forall \sigma''_j \in H_m : \mathcal{J}(\sigma''_j) = \sigma_{n+j} \in H_{n+m}, \quad 1 \le j \le m-1. \end{aligned} \tag{2.14}$$

By construction $\mathcal{J}(H_n)$ and $\mathcal{J}(H_m)$ form two mutually commuting Hecke subalgebras in H_{m+n}.

Let $\lambda \vdash n$, $\mu \vdash m$ and $M'_\lambda(i)$, $M''_\mu(j)$ be two right submodules in H_n and H_m generated by the primitive idempotents $Y^\lambda_{ii}(\sigma')$ and $Y^\mu_{jj}(\sigma'')$ correspondingly. The image of the tensor product $M'_\lambda(i) \otimes M''_\mu(j)$ under (2.14) is a right $H_n \times H_m$ module $M^r_\lambda(i) \otimes M^r_\mu(j) \subset H_{n+m}$ and obviously

$$V_\lambda(i) \otimes V_\mu(j) = \mathrm{Im}\, \rho_R(M^r_\lambda(i) \otimes M^r_\mu(j)). \tag{2.15}$$

The right H_{n+m} module induced from $M^r_\lambda(i) \otimes M^r_\mu(j)$ is reducible and can be decomposed into a direct sum of irreducible right H_{n+m} submodules $M^r_\nu(k)$. This decomposition is the

[10] The detailed description of Schur functions and related topics can be found in [16].

image (under monomorphism $M^r_\lambda \to M^r_\lambda(i)$) of the following relation between irreducible modules M^r_λ and M^r_μ:

$$M^r_\lambda \otimes M^r_\mu = c^\nu_{\lambda\mu} M^r_\nu, \tag{2.16}$$

where the coefficients $c^\nu_{\lambda\mu}$ are equal to the multiplicity of irreducible H_{n+m} characters χ^ν in the character induced from $\chi^\lambda \times \chi^\mu$. Due to (2.15) the coefficients $c^\nu_{\lambda\mu}$ will also determine the expansion of tensor product of two basic objects V_λ and V_μ.

Now we use the fact that at generic q the Hecke algebra H_m is isomorphic to the group algebra $\mathbb{K}(S_m)$ of the mth-order permutation group S_m for all $m \geq 1$. Therefore, the coefficients $c^\nu_{\lambda\mu}$ defining the multiplicity of irreducible module M^r_ν in the tensor product $M^r_\lambda \otimes M^r_\mu$ are the same for the Hecke algebra and for the algebra $\mathbb{K}(S_m)$. To complete the proof, note that as is well known from the representation theory of symmetric group (see, e.g., [16]) the corresponding multiplicities coincide with the Littlewood–Richardson coefficients in the product of symmetric Schur functions s_λ. □

Remark 7. The explicit calculation of the decomposition of $M^r_\lambda(i) \otimes M^r_\mu(j)$ into a direct sum of H_{n+m} submodules gives

$$M^r_\lambda(i) \otimes M^r_\mu(j) = \alpha^\nu_{\lambda\mu}(k(i,j)) M^r_\nu(k(i,j)), \quad \alpha^\nu_{\lambda\mu}(k(i,j)) \in \mathbb{Z}_+.$$

But the isomorphic submodules $M^r_\nu(k)$ with different k are images of the same module M^r_ν equipped with different monomorphisms (depending on i and j) $M^r_\nu \to \bar{M}_\nu \subset H_{n+m}$. The sum $\sum_k \alpha^\nu_{\lambda\mu}(k)$ does not depend on i and j and is equal to $c^\nu_{\lambda\mu}$ which defines the structure of the tensor product (2.16).

Now we should convert the monoidal category SW(V) into a quasitensor one (see [3]). This means, that we need to define a set of *natural commutativity* (*or braiding*) *isomorphisms* (1.2) for any pair $U, V \in \mathrm{Ob}(\mathrm{SW(V)})$. To be compatible with the monoidal structure the braiding isomorphisms must satisfy the following property for any triple $U, V, W \in \mathrm{Ob}(\mathrm{SW(V)})$:

$$\begin{aligned} R_{U,V\otimes W} &= (\mathrm{id}_V \otimes R_{U,W}) \circ (R_{U,V} \otimes \mathrm{id}_W), \\ R_{V\otimes W,U} &= (R_{V,U} \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes R_{W,U}). \end{aligned} \tag{2.17}$$

Let us recall that the associativity isomorphisms are taken to be identical.

Besides, considering two possible ways of transformation of $U \otimes V \otimes W$ into $W \otimes V \otimes U$ one comes to the following condition:

$$\begin{aligned} &(R_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes R_{U,W}) \circ (R_{U,V} \otimes \mathrm{id}_W) \\ &\quad = (\mathrm{id}_W \otimes R_{U,V}) \circ (R_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes R_{V,W}). \end{aligned} \tag{2.18}$$

Let us begin with the simplest object—the space V. As a braiding isomorphism for $V \otimes V$ we take the Hecke symmetry R. Requirement (2.18) on $V^{\otimes 3}$ transforms into the Yang–Baxter equation (1.7) on R which is satisfied by the definition of R. The decomposition property (2.17) will be used as *a definition* of braiding isomorphisms for an arbitrary tensor power

of the space V. That is we take:

$$R_{V,V^{\otimes k}} = R_k R_{k-1} \cdots R_1, \qquad R_{V^{\otimes k},V} = R_1 R_2 \cdots R_k. \tag{2.19}$$

In the above formula a shorthand notation $R_i \equiv R_{ii+1}$ is used. Introducing one more notation for the chain of R matrices

$$R_{i\to j} \equiv \begin{cases} R_i R_{i+1} \cdots R_j & \text{if } \ j = i+n \geq i \\ R_i R_{i-1} \cdots R_j & \text{if } \ j = i-n \leq i \end{cases}$$

we can write a compact form for the braiding isomorphism in the general case:

$$\begin{aligned} R_{V^{\otimes n},V^{\otimes m}} &= R_{m\to(m+n-1)} R_{(m-1)\to(m+n-2)} \cdots R_{1\to n} \\ &\equiv R_{m\to 1} R_{(m+1)\to 2} \cdots R_{(n+m-1)\to n}. \end{aligned} \tag{2.20}$$

The two parts of this formula correspond to two possible ways of passing from $V^{\otimes n} \otimes V^{\otimes m}$ to $V^{\otimes m} \otimes V^{\otimes n}$. Note, that all mappings (2.20) are morphisms of the first kind.

As for the braiding isomorphisms for $\mathbb{K} \otimes V$ we take it to be the usual flip:

$$\mathbb{K} \otimes V = V \otimes \mathbb{K} = V. \tag{2.21}$$

By definition of the objects of the category SW(V) any tensor power $V^{\otimes m}$ can be decomposed into a direct sum of basic objects V_λ, $\lambda \vdash m$. Therefore by making use of formula (2.20) one can define the braiding isomorphism for the tensor product of two arbitrary basic objects $V_\lambda \otimes V_\mu$, (and, therefore, for the tensor product of any couple of objects) if we manage to prove that isomorphism (2.20) does not "destroy" the structure of embeddings of $V_\lambda \otimes V_\mu$ into $V^{\otimes n} \otimes V^{\otimes m}$. If it is the case then we can take the restriction of (2.20) onto $\bar{V}_\lambda \otimes \bar{V}_\mu$ as the braiding morphism for the tensor product of V_λ and V_μ.

That is we have to show the following. Consider two arbitrary embeddings $\mathcal{J}_i : V_\lambda \hookrightarrow V^{\otimes n}$ and $\mathcal{J}_j : V_\mu \hookrightarrow V^{\otimes m}$. Let $\mathcal{J}_i(V_\lambda) = V_\lambda(i)$ and $\mathcal{J}_j(V_\mu) = V_\mu(j)$. Let us recall that $V_\lambda(i)$ is defined by (2.11). We want to show that under isomorphism (2.20) one gets:

$$V_\lambda(i) \otimes V_\mu(j) \to R_{\lambda\mu}(V_\lambda(i) \otimes V_\mu(j)),$$

where $R_{\lambda\mu}$ is an invertible operator from $\mathrm{End}(\bar{V}_\lambda \otimes \bar{V}_\mu)$ (the space $\bar{V}_\lambda$ is defined in (2.12)) which *does not* depend on i and j.

Proposition 8. *For two given partitions $\lambda \vdash n$, $\mu \vdash m$ and two arbitrary integers $1 \leq i \leq d_\lambda$ and $1 \leq j \leq d_\mu$ consider the corresponding spaces $V_\lambda(i)$ and $V_\mu(j)$ as defined in (2.11). Then under isomorphism (2.20) one has*

$$V_\lambda(i) \otimes V_\mu(j) \to R_{\lambda\mu}(V_\lambda(i) \otimes V_\mu(j)), \tag{2.22}$$

where the operator $R_{\lambda\mu}$ is defined on $\bar{V}_\lambda \otimes \bar{V}_\mu$ by means of the following formula:

$$R_{\lambda\mu} = (\rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda)) \cdot R_{V^{\otimes n},V^{\otimes m}}. \tag{2.23}$$

Here $R_{V^{\otimes n},V^{\otimes m}}$ is defined by (2.20) and

$$\mathbb{Y}^\lambda \equiv \sum_{i=1}^{d_\lambda} Y^\lambda_{ii}$$

is a central idempotent of the corresponding Hecke algebra.

Proof. The proof is based on the embedding of $H_n \times H_m$ into H_{n+m} introduced in Proposition 6. In accordance with (2.14) and (2.15) we can write

$$V_\lambda(i) \otimes V_\mu(j) = \mathrm{Im}\{\rho_R(Y^\lambda_{ii}) \otimes \rho_R(Y^\mu_{jj})\}.$$

Considered on the arbitrary basis element of the space $V^{\otimes n} \otimes V^{\otimes m}$ the right-hand side of the above formula reads:

$$e_{\langle 1|} \otimes \cdots \otimes e_{\langle n+m|} \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle n+m|}\{\mathcal{Y}^\lambda_{ii} \otimes \mathcal{Y}^\mu_{jj}\}^{|1\cdots n+m\rangle}_{\langle 1\cdots n+m|}, \tag{2.24}$$

where the matrices $\mathcal{Y}^\lambda_{ii} = \rho_R(Y^\lambda_{ii})$ and $\mathcal{Y}^\mu_{jj} = \rho_R(Y^\mu_{jj})$ are some polynomials in $R_1 \cdots R_{n-1}$ and $R_{n+1} \cdots R_{n+m-1}$, respectively:

$$\mathcal{Y}^\lambda_{ii} = \mathcal{Y}^\lambda_{ii}(R_1, \ldots, R_{n-1}), \qquad \mathcal{Y}^\mu_{jj} = \mathcal{Y}^\mu_{jj}(R_{n+1}, \ldots, R_{n+m-1}).$$

Applying the braiding isomorphism (2.20) to $V^{\otimes n} \otimes V^{\otimes m}$ leads to the following result for formula (2.24):

$$e_{\langle 1|} \otimes \cdots \otimes e_{\langle n+m|} \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle n+m|}\{R_{V^{\otimes n}, V^{\otimes m}} \cdot (\mathcal{Y}^\lambda_{ii} \otimes \mathcal{Y}^\mu_{jj})\},$$

where the symbol $\cdot$ stands for the matrix multiplication.

Now one should take into account the following relations which are direct consequence of (2.20) and Yang–Baxter equation (1.7):

$$R_{V^{\otimes n}, V^{\otimes m}} \cdot R_i = R_{i+m} \cdot R_{V^{\otimes n}, V^{\otimes m}}, \quad 1 \le i \le n-1,$$
$$R_{V^{\otimes n}, V^{\otimes m}} \cdot R_j = R_{j-n} \cdot R_{V^{\otimes n}, V^{\otimes m}}, \quad n+1 \le j \le n+m-1.$$

Using these relations one gets:

$$\begin{aligned} &R_{V^{\otimes n}, V^{\otimes m}} \cdot \mathcal{Y}^\lambda_{ii}(R_1 \cdots R_{n-1})\mathcal{Y}^\mu_{jj}(R_{n+1} \cdots R_{n+m-1}) \\ &\quad = \mathcal{Y}^\mu_{jj}(R_1 \cdots R_{m-1}) \otimes \mathcal{Y}^\lambda_{ii}(R_{m+1} \cdots R_{m+n-1}) \cdot R_{V^{\otimes n}, V^{\otimes m}}. \end{aligned}$$

This formula proves (2.22). In order to find the form of $R_{\lambda\mu}$ we observe that $\mathbb{Y}^\lambda = \sum_i Y^\lambda_{ii}$ are *central* elements of $H_n(\lambda \vdash n)$ and besides $Y^\lambda_{ii} \equiv Y^\lambda_{ii}\mathbb{Y}^\lambda$ $\forall i, j$. Therefore

$$(\mathcal{Y}^\mu_{jj} \otimes \mathcal{Y}^\lambda_{ii}) \cdot R_{V^{\otimes n}, V^{\otimes m}} \equiv \mathcal{Y}^\mu_{jj} \otimes \mathcal{Y}^\lambda_{ii} \cdot (\rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda) \cdot R_{V^{\otimes n}, V^{\otimes m}}),$$

where $\rho_R(\mathbb{Y}^\lambda)$ is the projector onto $\bar{V}_\lambda \subset V^{\otimes(n+m)}$. Thus, we come to form (2.23) of $R_{\lambda\mu}$. □

So, the operator $R_{\lambda\mu} = \rho_R(\mathbb{Y}^\mu) \otimes \rho_R(\mathbb{Y}^\lambda) R_{V^{\otimes n}, V^{\otimes m}}$ does not depend on a concrete embedding of $V_\lambda \otimes V_\mu \in \mathrm{Ob}(\mathrm{SW}(\mathrm{V}))$ into $V^n \otimes V^m$ (i.e., it does not depend on indices i, j) and represents a braiding isomorphism for $V_\lambda \otimes V_\mu$.

2.3. *Morphisms of the second kind and reduction procedure*

As was mentioned above (see (2.6)) the image of the antisymmetrizer A^m with $m > p$ is identical zero in $V^{\otimes m}$. It can be shown that the same is true for any $\mathrm{Im}\, \rho_R(M^r_\lambda)$ in case

if the height $\ell(\lambda) > p$. This means that the basic objects of our category V_λ are labeled by partitions with restricted height: $V_\lambda \not\equiv 0 \Leftrightarrow \ell(\lambda) \leq p$.

However, there is another consequence of (2.5) and (2.6) which concerns the objects labeled by partitions with the height equal to p. Consider a partition $\lambda \vdash (p+m)$ for some nonnegative integer m such that $\ell(\lambda) = p$. Let μ denote a partition of m which is obtained from λ by striking the first column out of the Young diagram corresponding to λ. The parts of λ and μ are connected by the relation $\mu_i = \lambda_i - 1\ \forall \lambda_i \neq 0$. As immediately follows from Proposition 6 and from the above remark about the maximal height of λ the following decomposition takes place:

$$V_\lambda \cong V_{(1^p)} \otimes V_\mu, \tag{2.25}$$

where $V_{(1^p)}$ stands for the one-dimensional space labeled by one-column diagram with p boxes: $\lambda = (1^p)$.

Isomorphisms (2.25) allow us to define the so-called *reduction procedure* and to introduce morphisms of the second kind in our category, namely, those identifying V_λ and V_μ. For this purpose consider a mapping ψ sending the one-dimensional space $V_{(1^p)}$ into the field $\mathbb{K}$. Evidently, one only needs to fix the action of the mapping ψ on a basis vector of $V_{(1^p)} \cong \operatorname{Im} \rho_R(A^{(p)})$. In the general form such a mapping looks like

$$\psi(e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} v^{|1\cdots p\rangle}) = 1 \in \mathbb{K}. \tag{2.26}$$

The choice of 1 in the above formula does not restrict generality of our consideration.

Let us suppose that $\psi \in \operatorname{Mor}(V_{(1^p)}, \mathbb{K})$. Then taking into account (1.3) and the fact that the identity operator is a categorical morphism we conclude that the following diagrams must be commutative:

$$\begin{array}{ccc} V_{(1^p)} \otimes V_\mu & \xrightarrow{\psi\otimes\mathrm{id}} & \mathbb{K} \otimes V_\mu \\ {\scriptstyle R_{V_{(1^p)},V_\mu}}\downarrow & & \| \\ V_\mu \otimes V_{(1^p)} & \xrightarrow{\mathrm{id}\otimes\psi} & V_\mu \otimes \mathbb{K} \end{array} \qquad \begin{array}{ccc} V_\mu \otimes V_{(1^p)} & \xrightarrow{\mathrm{id}\otimes\psi} & V_\mu \otimes \mathbb{K} \\ {\scriptstyle R_{V_\mu,V_{(1^p)}}}\downarrow & & \| \\ V_{(1^p)} \otimes V_\mu & \xrightarrow{\psi\otimes\mathrm{id}} & \mathbb{K} \otimes V_\mu. \end{array}$$

Consider a particular case of these diagrams by putting $V_\mu = V$ (if they are commutative in this case the same will be true for any μ). Then by passing to the basis we have

$$V_{(1^p)} \otimes V: \quad e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} \otimes e_{\langle p+1|} v^{|1\cdots p\rangle} \xrightarrow{\psi\otimes\mathrm{id}} 1 \otimes e_{\langle p+1|} = e_{\langle p+1|} \otimes 1. \tag{2.27}$$

According to another way in the left diagram we should first apply the braiding morphism $R_{V_{(1^p)},V}$. Thus, we have

$$\begin{aligned} &R_{V_{(1^p)},V}: e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} \otimes e_{\langle p+1|} v^{|1\cdots p\rangle} \\ &\quad \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} \otimes e_{\langle p+1|} R_1 \cdots R_p\, v^{|1\cdots p\rangle}. \end{aligned}$$

In order to simplify this expression we need the following useful relations (see [13]):

$$\begin{aligned} &R_1 \cdots R_p \mathcal{A}^{(p)} = (-1)^{p-1} q p_q \mathcal{A}^{(2,p+1)} \mathcal{A}^{(p)}, \\ &R_p \cdots R_1 \mathcal{A}^{(2,p+1)} = (-1)^{p-1} q p_q \mathcal{A}^{(p)} \mathcal{A}^{(2,p+1)}, \\ &\mathcal{A}^{(p)} \mathcal{A}^{(2,p+1)} \mathcal{A}^{(p)} = p_q^{-2} \mathcal{A}^{(p)} \otimes \mathrm{id}_{p+1}. \end{aligned} \tag{2.28}$$

In these formulas the antisymmetrizer $\mathcal{A}^{(2,p+1)}$ has the same form as $\mathcal{A}^{(p)}$ but it depends on $R_2, \ldots, R_p$ instead of $R_1, \ldots, R_{p-1}$.

Now in virtue of (2.28) we obtain:

$$R_1 \cdots R_p v^{|1\cdots p\rangle} = v^{|2\cdots p+1\rangle} N^{|1\rangle}_{\langle p+1|}, \tag{2.29}$$

where the matrix N is defined as follows:

$$N^i_j \equiv (-1)^{p-1} q p_q u_{a_2\cdots a_p j} v^{i a_2 \cdots a_p} \quad \text{or} \quad N^{|1\rangle}_{\langle 1'|} = (-1)^{p-1} q p_q u_{\langle 2\cdots p 1'|} v^{|12\cdots p\rangle}. \tag{2.30}$$

Therefore the composition $(\mathrm{id} \otimes \psi) \circ R_{V_{(1^p)},V}$ gives the following result:

$$\begin{aligned} & e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} \otimes e_{\langle p+1|} v^{|1\cdots p\rangle} \\ & \quad \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle p|} \otimes e_{\langle p+1|} v^{|2\cdots p+1\rangle} N^{|1\rangle}_{\langle p+1|} \to e_{\langle 1|} N^{|1\rangle}_{\langle p+1|} \otimes 1, \end{aligned}$$

which obviously differs from (2.27). These results can be made compatible iff N is a scalar matrix (i.e., it is proportional to the $n \times n$ unit matrix I). If it is so, then by multiplying the braiding R by an appropriate factor we can achieve the commutativity of the left diagram above. Indeed, if we replace the braiding R by aR, $a \in \mathbb{K}$, the factor a^p appears in the right-hand side of (2.29). Choosing properly the factor a we can obtain $a^p N = I$.

Applying the same procedure for the space $V \otimes V_{(1^p)}$ (the right diagram above) we find:

$$e_{\langle 1|} \otimes e_{\langle 2|} \otimes \cdots \otimes e_{\langle p+1|} v^{|2\cdots p+1\rangle} \to 1 \otimes e_{\langle p+1|} M^{|p+1\rangle}_{\langle 1|},$$

where[11]

$$M^i_j \equiv (-1)^{p-1} q p_q u_{j a_2 \cdots a_p} v^{a_2 \cdots a_p i} \quad \text{or} \quad M^{|1'\rangle}_{\langle 1|} \equiv (-1)^{p-1} q p_q u_{\langle 12\cdots p|} v^{|2\cdots p 1'\rangle}. \tag{2.30$'$}$$

For the same reason we assume the matrix M to be scalar. Then the right diagram can also be made commutative by a proper rescaling of R. However, if we want to obtain the unit matrix instead of the matrices N and M simultaneously we have to impose one more condition: $M = N$.

Therefore from now on we will suppose the Hecke symmetry R to satisfy the relation

$$M = N = aI, \quad a \in \mathbb{K}^\times. \tag{2.31}$$

Thus, the mapping ψ becomes a morphism of the category SW(V) after a proper renormalization of the braiding morphisms.

This normalization factor is easy to find. Indeed, using relations (2.28) (valid for *any* Hecke symmetry of the rank p) we find

$$M \cdot N = q^2 I. \tag{2.32}$$

So, if relation (2.31) is satisfied then by virtue of (2.32) we have $a = \pm q$ and hence

$$N = M = \pm qI. \tag{2.33}$$

[11] Note that the matrices N and M differ by a factor from those considered in [10].

Therefore by assuming ψ to be a morphism and the braiding to be natural we should pass from the initial Hecke symmetry R to the braiding

$$\bar{R} = (\pm q)^{-1/p} R. \tag{2.34}$$

On the higher tensor powers this renormalization gives rise to a renormalized braiding as well

$$\bar{R}_{V^{\otimes n}, V^{\otimes m}} = (\pm q)^{-mn/p} R_{V^{\otimes n}, V^{\otimes m}}. \tag{2.35}$$

The mapping ψ (as well as its inverse ψ^{-1}) will be called a morphism of the second kind.

Now let us compare our approach to introducing categorical morphisms with that making use of the RTT algebra. Recall that this algebra denoted $\mathcal{T}$ is generated by n^2 elements T^i_j subject to the relations [8]:

$$R_{12} T_1 T_2 = T_1 T_2 R_{12}. \tag{2.36}$$

It becomes a bialgebra being equipped with a coproduct Δ and a counit ε as follows:

$$\Delta(T^i_j) = T^i_k \otimes T^k_j, \qquad \varepsilon(T^i_j) = \delta^i_j.$$

Define the right comodule structure $\delta_r : V \to V \otimes \mathcal{T}$ on the space V as follows:

$$\delta_r(v) = e_k \otimes T^k_i v^i \quad \forall\, v = v^i e_i \in V.$$

Such a coaction is extended to $V^{\otimes k}$ in the obvious way:

$$\delta_r : V^{\otimes k} \to V^{\otimes k} \otimes \mathcal{T}, \qquad e_{\langle 1|} \otimes \cdots \otimes e_{\langle k|} \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle k|} \otimes T_1 \cdots T_k, \tag{2.37}$$

and all the comodule properties are easily verified. Then each $V_\lambda(i)\lambda \vdash k$ turns out to be an invariant subcomodule in $V^{\otimes k}$ since the coaction δ_r commute with the action of $\rho_R(M^r_\lambda(i))$ on $V^{\otimes k}$:

$$\forall k, \quad \forall \lambda \vdash k, \quad \rho_R(M^r_\lambda) T_1 T_1 \cdots T_k = T_1 T_1 \cdots T_k \rho_R(M^r_\lambda).$$

This equation is a direct consequence of defining relations (2.36) and explicit formula for primitive idempotents of H_k which expresses each Y^λ_{ii} as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$.

Thus, any object of the category SW(V) can be equipped with a $\mathcal{T}$-coaction. This structure is often used in order to define categorical morphisms: one says that a map $U \to V$ is a morphism if it commutes with this coaction. Let us analyze what it entails being applied to ψ. For this purpose consider *a quantum determinant* $\det_q T$ of bialgebra (2.36) defined as follows [8]:

$$\det_q T \equiv u_{\langle 12\cdots p|} T_1 T_2 \cdots T_p v^{|12\cdots p\rangle}. \tag{2.38}$$

The commutation relations of $\det_q T$ with generators of (2.36) read (cf. [10]):

$$\det_q T \cdot T = (N^{-1} T N) \cdot \det_q T \Leftrightarrow \det_q T \cdot (NT) = (TN) \cdot \det_q T \tag{2.39}$$

or due to (2.32)

$$\det_q T \cdot T = (MTM^{-1}) \cdot \det_q T \Leftrightarrow \det_q T \cdot (TM) = (MT) \cdot \det_q T.$$

If we want the mapping ψ (2.26) to commute with the comodule structure we have to introduce one more condition:

$$\det_q T = 1. \tag{2.40}$$

But such a relation is compatible with the algebraic structure only iff $\det_q T$ is a central element of algebra (2.36). By virtue of (2.39) this means that the matrix N (and hence M) must be scalar.

Formally, the condition "N and M are scalar" is weaker than "N and M are scalar and equal to each other". It is not clear whether there exist Hecke symmetries such that the matrices N and M are scalar but not equal to each other. Nevertheless, the family of even Hecke symmetries satisfying (2.33) is sufficiently large. It will be shown by one of the authors (RL) in a separate paper.

Let us also remark that for the quasiclassical Hecke symmetries coming from the universal R matrix of QG $U_q(sl(n))$ condition (2.31) holds true. The quantum determinant $\det_q T$ and the unit element 1 generate the center of (2.36) [8]. But in general (even in the quasiclassical case) $\det_q T$ is not central.

2.4. The structure of rigid category

To convert a (quasi)tensor category $\mathcal{C}$ into a *rigid* one we must specify the following data [3]:

a. A mapping $* : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ which for any $U \in \mathrm{Ob}(\mathcal{C})$ put into correspondence its left dual U^*
b. For any pair of dual objects U and U^* there exist evaluation $\mathbf{ev}_U$ and coevaluation $\boldsymbol{\pi}_U$ morphisms of $\mathcal{C}$

$$\boldsymbol{\pi}_U : \mathbb{K} \to U \otimes U^*, \qquad \mathbf{ev}_U : U^* \otimes U \to \mathbb{K},$$

such that the following diagrams are commutative:

$$\begin{array}{ccc} U & \xrightarrow{\boldsymbol{\pi}_U \otimes \mathrm{id}} & (U \otimes U^*) \otimes U \\ \mathrm{id}\downarrow & & \downarrow\mathrm{id} \\ U & \xleftarrow{\mathrm{id}\otimes \mathbf{ev}_U} & U \otimes (U^* \otimes U) \end{array} \qquad \begin{array}{ccc} U^* & \xrightarrow{\mathrm{id}\otimes \boldsymbol{\pi}_U} & U^* \otimes (U \otimes U^*) \\ \mathrm{id}\downarrow & & \downarrow\mathrm{id} \\ U^* & \xleftarrow{\mathbf{ev}_U \otimes \mathrm{id}} & (U^* \otimes U) \otimes U^*. \end{array} \tag{2.41}$$

Let us begin with the case $U = V$. By our construction of the morphism ψ it is evident that the dual space V^* is nothing but $\Lambda^{p-1}(V)$. So, in order to satisfy (2.41) we should only fix a convenient basis converting $\Lambda^{p-1}(V)$ into a left dual space.

Definition 9. The space $\Lambda^{p-1}(V)$ equipped with the basis

$$e^i = e_{a_2} \otimes \cdots \otimes e_{a_p} v^{ia_2\cdots a_p} \quad \text{or} \quad e^{|1\rangle} = e_{\langle 2|} \otimes \cdots \otimes e_{\langle p|} v^{|12\cdots p\rangle} \tag{2.42}$$

will be called a left dual space to the space V. The morphisms $\mathbf{ev}_V$ and $\boldsymbol{\pi}_V$ are defined as follows (on basis vectors):

$$\mathbf{ev}_V : e^i \otimes e_j \to \delta^i_i 1, \qquad \boldsymbol{\pi}_V : 1 \to e_i \otimes e^i. \tag{2.43}$$

In order to justify this definition we have to show the following.

Proposition 10. *Mappings (2.43) are morphisms of the SW(V) category and they satisfy property (2.41).*

Proof. The fact that $\boldsymbol{\pi}_V$ is a morphism is evident since

$$\boldsymbol{\pi}_V = \psi^{-1}.$$

To show that $\mathbf{ev}_V$ is a morphism let us consider the mapping

$$\varphi = (-1)^{p-1} p_q \psi \circ \mathcal{A}^p : \quad V^{\otimes p} \to \mathbb{K}.$$

It is a morphism by construction. In a basis form we get

$$\varphi : e_{i_1} \otimes \cdots \otimes e_{i_p} \to (-1)^{p-1} p_q u_{i_1 \cdots i_p}.$$

Applying this morphism to the element

$$e^i \otimes e_j = v^{ia_2 \cdots a_p} e_{a_2} \otimes \cdots \otimes e_{a_p} \otimes e_j,$$

we get

$$\varphi(e^i \otimes e_i) = (-1)^{p-1} p_q v^{ia_2 \cdots a_p} u_{a_2 \cdots a_p j} = q^{-1} N^i_j = \pm \delta^i_j.$$

Thus, the mapping $\mathbf{ev}_V$ is nothing but the morphism $\pm\varphi$ restricted on $\Lambda^{p-1}(V) \otimes V$.

Diagrams (2.41) are obviously commutative. □

Let us define now left duals to other simple objects of our category. First, introduce a dual object to $V^{\otimes m}$ by putting $(V^{\otimes m})^* = (V^*)^{\otimes m}$.

Proposition 11. *The space $(V^*)^{\otimes m}$ is dual to the space $(V^{\otimes m})$ being equipped with the mappings*

$$\mathbf{ev}_U : V^{*\otimes m} \otimes V^{\otimes m} \to \mathbb{K}, \qquad \boldsymbol{\pi}_U : \mathbb{K} \to V^{\otimes m} \otimes V^{*\otimes m},$$

which are defined to be:

$$\begin{aligned} &\mathbf{ev}_U : e^{|m\rangle} \otimes \cdots \otimes e^{|1\rangle} \otimes e_{\langle 1|} \otimes \cdots e_{\langle m|} \to \delta^{|1\rangle}_{\langle 1|} \cdots \delta^{|m\rangle}_{\langle m|}, \\ &\boldsymbol{\pi}_U : 1_{\mathbb{K}} \to e_{\langle 1|} \otimes \cdots \otimes e_{\langle m|} \otimes e^{|m\rangle} \otimes \cdots \otimes e^{|1\rangle}. \end{aligned} \tag{2.44}$$

Proof. It is obvious. □

Now we are ready to define the dual V^*_λ and corresponding morphisms $\boldsymbol{\pi}_{V_\lambda}$ and $\mathbf{ev}_{V_\lambda}$ for arbitrary basic object V_λ of the category SW(V).

Fix a basic object V_λ, $\lambda \vdash m$, and consider its particular embedding into $V^{\otimes m}$ in the form (2.11):

$$V_\lambda(i) = \operatorname{Im} \rho_R(Y^\lambda_{ii}).$$

This space is spanned by the following vectors

$$e_{\langle 1|} \otimes \cdots \otimes e_{\langle m|} \cdot (\mathcal{Y}^\lambda_{ii})^{|1\cdots m\rangle}_{\langle 1\ldots m|},$$

where $\mathcal{Y}^\lambda_{ii}$ is a matrix of Y^λ_{ii} in the representation ρ_R.

Let us define a space $V^*_\lambda(i) \subset V^{*\otimes m}$ as a linear span of the form:

$$V^*_\lambda(i) = \{w^* : w^* = w_{\langle 1\cdots m|}(\mathcal{Y}^\lambda_{ii}) \cdot e^{|m\rangle} \otimes \cdots \otimes e^{|1\rangle}\}, \tag{2.45}$$

where $w_{\langle 1\cdots m|}$ is an arbitrary tensor with components from $\mathbb{K}$. It is important, that the order of spaces in this formula is reversed comparing with the preceding relation while the matrix $\mathcal{Y}^\lambda_{ii}$ is the same.

Finally, we have the following proposition.

Proposition 12. *The space V^*_λ being equipped with the morphisms $\mathbf{ev}_{V_\lambda}$ and $\boldsymbol{\pi}_{V_\lambda}$ of the form*

$$\mathbf{ev}_{V_\lambda} : V^*_\lambda \otimes V_\lambda \to \mathcal{Y}^\lambda_{ii}, \qquad \boldsymbol{\pi}_{V_\lambda} : 1_{\mathbb{K}} \to e_{\langle 1|} \otimes \cdots e_{\langle m|} \otimes \mathcal{Y}^\lambda_{ii} \cdot e^{|m\rangle} \otimes \cdots \otimes e^{|1\rangle} \tag{2.46}$$

becomes left dual to the space V_λ (provided that V_λ is realized as $V_\lambda(i)$).

Let us note that in fact the space V^*_λ is nothing but

$$V_\mu, \quad \mu = (\lambda_1, \lambda_1 - \lambda_{p-1}, \lambda_1 - \lambda_{p-2}, \ldots, \lambda_1 - \lambda_2)$$

equipped with an embedding in the space $V^{*\otimes m}$. This shows that we do not need to construct left dual to the space V^*_λ because it is just the object V_λ.

Nevertheless, the problem of an explicit pairing

$$V_\lambda \otimes V^*_\lambda \to \mathbb{K}$$

converting V^*_λ into the right dual to V_λ (and V_λ in the left dual to V^*_λ) is very important. In order to describe this pairing in a particular case $V_\lambda = V$ let us calculate the braiding of V and $\Lambda^{p-1}(V)$ expressed via the basis $\{e^i\}$. Note, that we use the renormalized braiding $\bar{R}$ (2.34).

Proposition 13. *We have*

$$\bar{R}_{V,V^*} : e_i \otimes e^j \to e^r \otimes e_s \bar{Q}^{js}_{ir}, \tag{2.47}$$

where in the given basis $\{e_i\}$ the operator $\bar{Q}$ satisfies the relation:

$$\bar{R}^{ia}_{jb} \bar{Q}^{bk}_{al} = \delta^i_l \delta^k_j \Leftrightarrow \bar{Q}^{ia}_{jb} \bar{R}^{bk}_{al} = \delta^i_l \delta^k_j. \tag{2.48}$$

Note, that if such $\bar{Q}$ exists then the corresponding braiding $\bar{R}$ is usually called *invertible by column*.

Proof. According to definition (2.42) for e^i and (2.35) for the renormalized braiding we get (omitting the obvious signs of tensor product):

$$e_i \otimes e^j = e_i e_{a_2} \cdots e_{a_p} v^{ja_2\cdots a_p}$$
$$\overset{\bar{R}_{V,V^{\otimes(p-1)}}}{\to} e_{s_1} \cdots e_{s_p} (\pm q)^{(1-p)/p} (R_{p-1} \cdots R_1)^{s_1\cdots s_p}_{ia_2\cdots a_p} v^{ja_2\cdots a_p}.$$

Now from (2.28) it is easy to find

$$(\pm q)^{(1-p)/p}(R_{p-1}\cdots R_1)^{s_1\cdots s_p}_{ia_2\cdots a_p}\, v^{ja_2\cdots a_p} = \bar{Q}^{js_p}_{ir}\, v^{rs_1\cdots s_{p-1}}.$$

Substituting this into the previous relation and applying again the definition (2.42) of the dual basis we find result (2.47). □

Now we introduce a pairing between V and V^* by putting

$$\overline{\mathbf{ev}}_V = \mathbf{ev}_V \circ R_{V,V^*}: \quad V \otimes V^* \to \mathbb{K}.$$

It is this operator which plays the central role in the next section where a categorical trace in the space $\mathrm{End}(V)$ will be introduced.

Completing this Section let us discuss a meaning of the property that the morphism ψ is coordinated with the braidings in the sense of (1.3). Let V be the basic object of the category SW(V). Assuming the category to be rigid we can identify $\mathrm{End}(V) \cong V \otimes \Lambda^{(p-1)}(V)$ (we consider the left morphisms space). Then the usual operator product $\mu : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)$ is nothing but the pairing $\Lambda^{(p-1)}(V) \otimes V \to \mathbb{K}$. Moreover, the property

$$R\mu_{12} = \mu_{23}R_{12}R_{23}, \qquad R\mu_{23} = \mu_{12}R_{23}R_{12}, \qquad R = R_{\mathrm{End}(V),\mathrm{End}(V)},$$

which appears in numerous applications is satisfied. Let us note that these relations are not invariant w.r.t. a rescaling $R \to aR$, $a \neq 0$ while the condition (2.40) is.

Also remark that often a problem arises to check that a given map $\rho : U \to V$, $U, V \in \mathrm{Ob}(\mathcal{C})$ is a categorical morphism. Then by putting in (1.3) $f = \rho$, $g = \mathrm{id}$ or $f = \rho$, $g = \rho$ we get necessary conditions very useful in practice.

3. Additive–multiplicative functional and dimension

In this section we find all possible additive and multiplicative (a–m) functionals on objects of SW category SW(V) and suggest a definition of the trace in $\mathrm{End}(V)$ as morphism (1.1) in the category such that the corresponding dimension is an a–m functional. As was pointed out in Section 1, the requirement that the trace should be a morphism in general is not a trivial one. In particular, this requirement gives rise to the notion of *quantum trace* in the category of finite-dimensional modules over a special class of quasitriangular Hopf algebras—the so-called ribbon Hopf algebras. In our approach the morphism property of trace can be obtained without any additional Hopf structure.

Let us recall that a mapping $f : \mathrm{Ob}(\mathrm{SW(V)}) \to \mathbb{K}$ will be called *an additive–multiplicative functional* (a–m functional) if it possesses the following property:

$$\forall U, V \in \mathrm{Ob}(\mathrm{SW(V)}): \quad f(U \oplus V) = f(U) + f(V), \quad f(U \otimes V) = f(U)f(V). \tag{3.1}$$

Remark 14. Introducing the Grothendieck semiring of the category SW(V) and making use of the fact that our category is semisimple (i.e., any of its object can be decomposed

into a direct sum of simple ones) we conclude that each a–m functional on the category is in fact defined on the Grothendieck semiring.

The fact that we are working with a Hecke symmetry of the rank p leads to an important consequence for the set of a–m functionals. Namely we will show that the value of any a–m functional f on a basic object V_λ (and, hence, on an arbitrary object of SW(V)) is completely determined by $(p-1)$ numbers[12] $f(V_{(1^k)})\, k = 1, \ldots, p-1$.

Proposition 15. *Given an a–m functional f, construct a pth-order polynomial in a formal variable t of the form:*

$$\phi(t) = t^p + f_1 t^{p-1} \cdots + f_{p-1} t + 1, \qquad f_k \equiv f(V_{(1^k)}), \quad k = 1, \ldots, p-1. \tag{3.2}$$

Let numbers $(-\alpha_i) \in \mathbb{K}$ be the roots of $\phi(t)$, that is $\phi(-\alpha_i) = 0\, i = 1, \ldots, p$. Then for any basic object V_λ of SW(V) the following relation holds:

$$f(V_\lambda) = s_\lambda(\alpha_1, \alpha_2, \ldots, \alpha_p), \tag{3.3}$$

where $s_\lambda(x_1, \ldots, x_p)$ is the symmetric Schur function in p variables[13] *corresponding to the partition λ.*

Proof. First of all let us prove that the numbers $f_k = f(V_{(1^k)})$ do satisfy (3.3). In accordance with the definition of α_i we can write

$$\phi(t) = (t + \alpha_1)(t + \alpha_2) \cdots (t + \alpha_p).$$

Therefore, as immediately follows from this relation, the coefficient f_k is the kth elementary symmetric function e_k in variables α_i [16]:

$$f_k = \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} = e_k(\alpha_1, \ldots, \alpha_k).$$

But since $s_{(1^k)} = e_k$ we conclude that the assertion of the proposition is valid for f_k.

The fact that the quantities $s_\lambda(\alpha_1, \alpha_2, \ldots, \alpha_p)$ can be considered as values of an a–m functional follows immediately from the properties of Schur functions and from Proposition 6. In order to prove (3.3) we should only check that once the quantities $f(V_\lambda)$, $\lambda = (1^k)$, are given then all quantities $f(V_\mu)$ for all other partitions are uniquely defined. This can be shown by induction in couples (m, k), where m is the number of columns in λ and k is the number of boxes in the last column. Namely, using the implication

$$f(V_\lambda \otimes V_\mu) = f(V_\lambda) f(V_\mu) = c^\nu_{\lambda\mu} f(V_\nu) \Rightarrow s_\lambda s_\mu = c^\nu_{\lambda\mu} f(V_\nu),$$

we can find the quantity $f(V_\lambda)$ for λ corresponding to a given couple (m, k) provided that all $f(V_\mu)$, where μ corresponds to the couples (l, r) such that $l < m$ or $l = m, r < k$ are already known. □

[12] Due to (2.21) and reduction morphism ψ (2.26) we have to put for any a–m functional $f(\mathbb{K}) = f(V_{(1^p)}) = 1$.

[13] Let us recall, that due to (2.21) all the partitions λ labeling the objects of our category has the height not greater than p therefore the right-hand side of Eq. (3.3) is correctly defined for any object of SW(V).

In what follows we will use this proposition for a proper definition of the categorical trace. It will be defined as such a morphism (1.1) that the corresponding dimension becomes an a–m functional. By virtue of the above proposition we should only find the quantities (3.3) for $\lambda = (1^k)$.

Describe now this construction in detail. Begin with the basic object V. Since the space V is finite-dimensional, we can identify $\mathrm{End}(V)$ and $V \otimes V^*$ in the usual way:

$$\forall \mathcal{F} \in \mathrm{End}(V) \leftrightarrow e_i \otimes F^i_j e^j \in V \otimes V^*,$$

F^i_j being a matrix of $\mathcal{F}$ in the basis $\{e_i\}$. Define the mapping $\mathrm{tr}_V : \mathrm{End}(V) \to \mathbb{K}$ as the following composition of morphisms:

$$\mathrm{tr}_V = \alpha \mathbf{ev}_V \circ \bar{R}_{V,V^*}. \tag{3.4}$$

Note, that tr_V is defined by the above relation up to an arbitrary nonzero factor α which will be specified later from the requirement that the corresponding dimension would be an a–m functional. In the fixed basis $\{e_i\}$ of V one can write this mapping in an explicit form. Taking $\bar{R}_{V,V^*}$ from Proposition 13 we find:

$$\forall \mathcal{F} \in \mathrm{End}(V): \quad e_i \to e_j F^j_i \Rightarrow \mathrm{tr}_V(\mathcal{F}) = \alpha\, \mathrm{Tr}(F \cdot C), \tag{3.5}$$

where the symbol Tr means the usual matrix trace and the $n \times n$ matrix C is defined to be

$$C^i_{\ j} := \sum_{a=1}^{n} Q^{ia}_{ja}, \qquad Q \equiv (\pm q)^{-1/p} \bar{Q}.$$

Here the matrix $\bar{Q}$ is taken from (2.48) and the factor $(\pm q)^{1/p}$ is included into α.

The matrix C has the following useful properties [12]:

$$R_{12} C_1 C_2 = C_1 C_2 R_{12}, \tag{3.6}$$

$$C^{|1\rangle}_{\langle 1'|} = \frac{p_q}{q^p} u_{\langle 1'23\cdots p|} v^{|123\cdots p\rangle} \Rightarrow \mathrm{Tr}\, C = \frac{p_q}{q^p}, \tag{3.7}$$

$$\mathrm{Tr}_{(2)} R_{12} C_2 = \mathrm{id}_{(1)}. \tag{3.8}$$

The trace $\mathrm{tr}_{V^{\otimes k}}$ in $\mathrm{End}(V^{\otimes k})$ is defined (also up to an arbitrary factor $\beta_{(k)}$) as the mapping $V^{\otimes k} \otimes V^{*\otimes k} \to \mathbb{K}$:

$$\mathrm{tr}_{V^{\otimes k}} = \beta_{(k)}\, \mathrm{tr}^{(1)}_V \circ \mathrm{tr}^{(2)}_V \cdots \circ \mathrm{tr}^{(k)}_V, \tag{3.9}$$

where $\mathrm{tr}^{(i)}_V : V^{\otimes i} \otimes V^{*\otimes i} \to V^{\otimes(i-1)} \otimes V^{*\otimes(i-1)}$ reads

$$\mathrm{tr}^{(i)}_V = \mathrm{id}^{\otimes(i-1)} \otimes \alpha^{-1}\, \mathrm{tr}_V \otimes \mathrm{id}^{\otimes(i-1)}.$$

For any basic object V_λ $\lambda \vdash m$ one can find the explicit form of the trace in $\mathrm{End}(V_\lambda) \cong \mathrm{End}(V_\lambda(i))$ using definitions (2.11) and (2.45) and property (3.6):

$$\forall \mathcal{F} \in \mathrm{End}(V_\lambda(i)): \quad \mathrm{tr}_{V_\lambda}(\mathcal{F}) = \beta_{(m)}\, \mathrm{Tr}_{(1\cdots m)}(F \cdot C_\lambda), \tag{3.10}$$

where $\mathrm{Tr}_{(1\cdots m)}$ is the the usual trace in the matrix space $\mathrm{Mat}_{n\times n}^{\otimes m}(\mathbb{K})$ and the matrix C_λ is of the form:

$$C_\lambda = \mathcal{Y}_{ii}^{\lambda}(R_1\cdots R_{m-1})C_1C_2\cdots C_m. \tag{3.11}$$

Remark 16. At the first sight the trace tr_{V_λ} depends on a concrete embedding $V_\lambda \to V_\lambda(i)$. But in fact our definition is invariant w.r.t. different embeddings of V_λ into $V^{\otimes m}$. Indeed, as was already noticed in the previous section, any primitive idempotents Y_{ii}^{λ} of the Hecke algebra H_m can be transformed into another idempotent Y_{jj}^{λ} (with the same λ) with the help of an invertible element $X(i,j|\lambda)$ from H_m. The image $\mathcal{X} = \rho_R(X(i,j|\lambda))$ of the element $X(i,j|\lambda)$ under the representation ρ_R (2.1) represents a first kind morphism of SW(V) and according to (3.6) the string of matrices $C_1\cdots C_m$ in (3.11) commute with $\mathcal{X}$. Therefore trace (3.10) is actually independent of the index i labeling the concrete embedding $V_\lambda \to V_\lambda(i)$.

Find now the conditions on arbitrary factors α and $\beta_{(k)}$ which would guarantee the dimensions defined via the above trace to be an a–m functional.

Let us note that although the dimension defined by the usual trace is an a–m functional such a trace is not a morphism of the category if the braiding differs from the flip.[14] This is the reason why we should modify the usual trace by means of the matrix C (and all its extensions C_λ) multiplied by a proper factor α (3.5).

The fact that the matrix C extends to $\mathrm{End}(V^{\otimes m})$ in a "group like" way

$$C \to C^{\otimes m}$$

ensures the conservation of the additive and multiplicative structure of the functional (3.9) restricted on the identity operator (what is just categorical dimension) iff we put

$$\beta_{(k)} = \alpha^k. \tag{3.12}$$

So, if we want the trace tr_{V_λ} to be a morphism and giving rise to an a–m functional we have the only free parameter α at our disposal. This parameter (a normalization of the trace) can be found from the following condition $\dim_q V_{(1^p)} = 1$ (see footnote 12). The following lemma plays the key role in finding such a normalization (the relation presented below was found in another but equivalent form in [14]).

Lemma 17. *For any Hecke symmetry of the rank p the following relation holds:*

$$\mathrm{Tr}_{(1\cdots k)}(\mathcal{A}^{(k)}C_1\cdots C_k) = q^{-pk}\frac{p_q!}{k_q!(p-k)_q!}, \quad 1 \le k \le p, \tag{3.13}$$

where the matrix $\mathcal{A}^{(k)}$ is the image of kth-order antisymmetrizer $A^{(k)}$ under the representation ρ_R (2.1).

Proof. For an arbitrary $m \in \mathbb{Z}$ introduce the auxiliary notation

[14] For example, if our category SW(V) is supplied with the comodule structure over the RTT Hopf algebra (2.36) then the usual trace is not compatible with the comodule structure.

$$\mathcal{R}_i(m) := R_i - \frac{q^m}{m_q},$$

where m_q is defined in (1.9). With this notation the matrix $\mathcal{A}^{(m)}(R_1, \ldots, R_{m-1})$ can be presented in the form:

$$\mathcal{A}^{(m)} = \frac{(-1)^{m-1}}{m_q}\mathcal{A}^{(m-1)}\mathcal{R}_{m-1}(m-1)\mathcal{R}_{m-2}(m-2)\cdots\mathcal{R}_1(1), \qquad \mathcal{A}^{(1)} := \mathrm{id}.$$

For the sake of shortness we introduce the notation:

$$\mathrm{Tr}_{q(1\cdots k)}X := \mathrm{Tr}_{(1\cdots k)}(XC_1\cdots C_k).$$

Now we prove (3.13) by induction. For $k = 1$ we get with the help of (3.7)

$$\mathrm{Tr}\, C = q^{-p}p_q,$$

which gives the base of induction. Suppose that the assertion of the lemma is valid up to some integer $k < p$. Then using the simple relations

$$\mathrm{Tr}_{q(k+1)}\mathcal{R}_k(k) = -q^{-p}\frac{(p-k)_q}{k_q}\mathrm{id}_{(k)},$$
$$\mathcal{A}^{(k)}\mathcal{R}_{k-1}(k-1)\cdots\mathcal{R}_1(1) = (-1)^{k-1}k_q\mathcal{A}^{(k)},$$

one can complete the induction

$$\begin{aligned}
\mathrm{Tr}_{q(1\cdots k+1)}\mathcal{A}^{(k+1)} &= \frac{(-1)^k}{(k+1)_q}\mathrm{Tr}_{q(1\ldots k+1)}(\mathcal{A}^{(k)}\mathcal{R}_k(k)\cdots\mathcal{R}_1(1))\\
&= \frac{(-1)^k}{(k+1)_q}\mathrm{Tr}_{q(1\cdots k)}(\mathcal{A}^{(k)}[\mathrm{Tr}_{q(k+1)}\mathcal{R}_k(k)]\mathcal{R}_{k-1}(k-1)\cdots\mathcal{R}_1(1))\\
&= (-1)^{k+1}q^{-p}\frac{(p-k)_q}{k_q(k+1)_q}\mathrm{Tr}_{q(1\cdots k)}(\mathcal{A}^{(k)}\mathcal{R}_{k-1}(k-1)\cdots\mathcal{R}_1(1))\\
&= q^{-p}\frac{(p-k)_q}{(k+1)_q}\mathrm{Tr}_{q(1\cdots k)}\mathcal{A}^{(k)} = \frac{q^{-p(k+1)}p_q!}{(k+1)_q!(p-k-1)_q!}. \qquad \square
\end{aligned}$$

This lemma implies that if we take $\alpha = q^p$ (which is equivalent to simple renormalization $C \to q^pC$ in all formulas for the trace) then we get $\dim_q V_{(1^p)} = 1$ and the categorical dimension related to the trace becomes an a–m functional well defined on objects of our category and hence on the corresponding Grothendieck semiring.

Let us summarize the above consideration in the following proposition.

Proposition 18. *For any basic object V_λ, $\lambda \vdash m$, of the category $SW(V)$ define the trace tr_{V_λ} as a morphism $\mathrm{End}(V_\lambda) \to \mathbb{K}$ of the form:*

$$\forall\, \mathcal{F} \in \mathrm{End}(V_\lambda): \quad \mathrm{tr}_{V_\lambda}(\mathcal{F}) = q^{pm}\,\mathrm{Tr}_{(1\cdots m)}(F\cdot C_\lambda).$$

Then such a trace is a morphism in the category SW(V) and the corresponding categorical dimension is an a–m functional. The dimension of any basic object V_λ is as follows:

$$\dim_q V_\lambda := \mathrm{tr}_{V_\lambda}(\mathrm{id}) = s_\lambda(q^{p-1}, q^{p-3}, \ldots, q^{1-p}), \tag{3.14}$$

s_λ being the Schur symmetric function (polynomial) in p variables.

Proof. To prove (3.14) we use Proposition 15 and Lemma 17. By virtue of the trace definition one gets from (3.13):

$$\dim_q V_{(1^k)} = q^{pk}\,\mathrm{Tr}_{q(1\cdots k)}\mathcal{A}^{(k)} = \frac{p_q!}{k_q!(p-k)_q!} \equiv f_k.$$

Therefore due to Proposition 15 we should find the roots of the polynomial $\phi(t)$ (3.2) with the above f_k. But as is well known the generating function for such f_k (which are q-binomial coefficients) is as follows:

$$E_q(t) := \prod_{k=0}^{p-1}(q^{2k+1-p} + t) \equiv \phi(t).$$

Therefore the roots of the polynomial $\phi(t)$ with the coefficients $f_k = \dim_q V_{(1^k)}$ are the numbers $(-q^{2k+1-p})$, $k = 0, 1, \ldots, p-1$ and result (3.14) follows now from Proposition 15. □

Remark 19. We want to complete the paper with the following observation concerning the Koszul complexes considered in [10]. As was shown there, the Poincaré series $P_\pm(t)$

$$P_\pm(t) := \sum_l \dim \Lambda_\pm^l t^l$$

of "symmetric" and "skewsymmetric" algebras $\Lambda_\pm(V)$ (see (1.10)) satisfy the relation

$$P_+(t)P(-t) = 1. \tag{3.15}$$

We would like to point out that if we replace the usual dimensions in formula (3.15) by categorical ones relation (3.15) will be still valid. Moreover, the same is true if we replace the dimensions by the values of any a–m functional. It is not surprising since this fact reflects the well known relation between elementary and complete symmetric functions. And any a–m functional is nothing but a specialization of symmetric functions.

Thus, without constructing any deformed Koszul complex we can obtain some numerical characteristics of quantum objects merely replacing the usual dimensions by their q-analogs (if we disregard the property of the quantum differential to be a morphism of the category).

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